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# CLOSED SETS, LIMIT POINTS AND CONTINUOUS FUNCTIONS

In a topological space $(X, \mathcal{T})$, recall that $A \subseteq X$ is closed if $X \setminus A$ is open.

**Prop:** Given $(X, \mathcal{T})$ as above,
(i) any arbitrary intersection of closed sets is closed
(ii) any finite union of closed sets is closed
(iii) $\phi$ and $X$ are closed.

**Eg:** In any metric space $(X, d)$, singleton sets are always closed. Also, sets of the form $\{y \in X : d(x,y) \leq \varepsilon\}$ for some $x \in X$, $\varepsilon > 0$ are closed. The set above is called the "closed $\varepsilon$-ball centered at $x$".

**Proof:** Exercise - singleton sets in metric spaces are closed.
$A = \{y \in X \mid d(x,y) \leq \varepsilon\}$.
Proof that $A$ is closed: $X \setminus A = \{y \in X \mid d(x,y) > \varepsilon\}$
Let $r = d(x,y)$. Note $r > \varepsilon$.
Let $k = \inf\{d(y,a) : d(x,a) \leq \varepsilon\}$.

Claim: $k > 0$.

**Proof:** Let $a$ be a point s.t. $d(x,a) \leq \varepsilon$.
Then Note $d(x,y) \leq d(x,a) + d(a,y)$
So $d(a,y) \geq d(x,y) - d(x,a)$
$\geq r - \varepsilon$ $[\because -d(x,a) \geq -\varepsilon]$
This shows $d(y,a) \geq r - \varepsilon$ $\forall$ such $a$
$\therefore k \geq r - \varepsilon > 0$.

Take $\delta = k/2$.

**Note:** $B(y, \delta) \cap \{a : d(x,a) \leq \varepsilon\} = \phi$

$B(y,\delta) \cap A = \phi$
$\Rightarrow B(y, \delta) \subseteq X \setminus A \Rightarrow X \setminus A$ is open.

The set $A = \{a : d(x,a) \leq \varepsilon\}$ is called the closed $\varepsilon$-ball centered at $x$.

**Defn:** Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$.
The "closure of $A$" is the intersection of all closed sets in $X$ containing $A$.
The "interior of $A$" is the union of all open sets in $X$ that are contained in $A$.
We use the notation $\mathrm{Cl}\,A$ and $\mathrm{Int}\,A$ for the closure and interior of $A$ resp.

**Prop:** $\mathrm{Int}\,A \subseteq A \subseteq \mathrm{Cl}\,A$ for all subsets $A$ of a topological space $(X, \mathcal{T})$.

**Proof:** Clear.

**Rem:** $\mathrm{Cl}\,A$ is a closed set.
$\mathrm{Int}\,A$ is an open set.
If $A$ is closed, then $\mathrm{Cl}\,A = A$.
If $A$ is open, $\mathrm{Int}\,A = A$.

**Eg:** 1. In $(\mathbb{R}, \mathcal{T}_d)$, $\mathrm{Cl}(a,b) = [a,b]$ and

[illegible]

if every neighbourhood $U$ of $x$, the intersection $U \cap A$ contains a point $y \neq x$.

The "boundary of $A$" is the set $\mathrm{Cl}\,A \cap \mathrm{Cl}(X \setminus A)$.

We let $L(A)$ and $\partial A$ denote the set of limit points and the boundary of $A$ respectively.

**Remark:** $A \setminus L(A)$ is the set of "isolated points" of $A$.

**Prop:** In the setting above $x \in L(A) \Leftrightarrow x \in \mathrm{Cl}(A \setminus \{x\})$.

**Proof:**
$\Rightarrow$ If $x \in L(A)$, then $\forall$ nbhd $U$ of $x$,
$U \cap A$ contains $A \ni y \neq x$
$\Rightarrow U \cap (A \setminus \{x\}) \neq \phi$
$\Rightarrow x \in \mathrm{Cl}(A \setminus \{x\})$

$\Leftarrow$ If $x \in \mathrm{Cl}(A \setminus \{x\})$, then $\forall$ nbhd $U$ of $x$,
$U \cap A \setminus \{x\} \neq \phi \Rightarrow \exists y \neq x$ s.t. $y \in U$
$\Rightarrow x \in L(A)$.

**Prop:** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then $\mathrm{Cl}\,A = A \cup L(A)$.

**Proof:** $\mathrm{Cl}(A) \subseteq A \cup L(A)$: $x \in \mathrm{Cl}(A) \Leftrightarrow \forall$ nbhd $U$ of $x$, $U \cap A \neq \phi$
If $x \notin A$, $\Rightarrow \forall$ such $U$, $U \cap (A \setminus \{x\}) \neq \phi$
$\Rightarrow x \in \mathrm{Cl}(A \setminus \{x\})$
$\Rightarrow x \in L(A)$.

$A \cup L(A) \subseteq \mathrm{Cl}(A)$: We know $A \subseteq \mathrm{Cl}\,A$.
If $x \in L(A)$, then $x \in \mathrm{Cl}(A \setminus \{x\})$ $\Rightarrow x \in \mathrm{Cl}(A \setminus \{x\}) \subseteq \mathrm{Cl}(A)$.
**Exercise:** If $P \subseteq Q$, then $\mathrm{Cl}\,P \subseteq \mathrm{Cl}\,Q$ $\Rightarrow x \in \mathrm{Cl}\,A$.

**Examples**

1. $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ in $(\mathbb{R}, d)$.
(i) $\mathrm{Cl}\,B = B \cup \{0\}$
(ii) $\mathrm{Int}\,B = \phi$
(iii) $L(B) = \{0\}$

**Proof:** $0 \in L(B)$: Take $(a,b) \ni 0$.
$\exists n$ s.t. $a < 0 < \frac{1}{n} < b$
$\Rightarrow (a,b) \cap B \neq \phi$

Given $x \notin B$, $\exists \delta > 0$ s.t. $(x-\delta, x+\delta) \cap B = \phi$.

(iv) $\partial B = B \cup \{0\}$

**Proof:**
$\partial B = \mathrm{Cl}(B) \cap \mathrm{Cl}(\mathbb{R} \setminus B)$
$= (B \cup \{0\}) \cap \mathrm{Cl}(\mathbb{R} \setminus B) = (B \cup \{0\}) \cap \mathbb{R} = B \cup \{0\}$

$\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R} \setminus B$. Every open interval contains a point of $\mathbb{R} \setminus \mathbb{Q}$. $\Rightarrow \mathrm{Cl}(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathrm{Cl}(\mathbb{R} \setminus B)$

2. $\mathbb{Q}$ in $(\mathbb{R}, d)$.
(1) $\mathrm{Cl}\,\mathbb{Q} = \mathbb{R}$
**Proof:** Every open interval intersects $\mathbb{Q}$.

(2) $\mathrm{Int}\,\mathbb{Q} = \phi$
**Proof:** Every open interval intersects $\mathbb{R} \setminus \mathbb{Q}$. Thus no such open interval lies in $\mathbb{Q}$.
(3) $L(\mathbb{Q}) = \mathbb{R}$
**Proof:** Given $x \in \mathbb{R}$, and $(a,b) \ni x$, consider $(a,b) \setminus \{x\} = (a,x) \cup (x,b)$, which intersects $\mathbb{Q}$
$\Rightarrow x \in \mathrm{Cl}(\mathbb{Q} \setminus \{x\})$ $\forall x \in \mathbb{R}$.
(4) $\partial \mathbb{Q} = \mathbb{R}$
**Proof:** $\mathrm{Cl}\,\mathbb{Q} = \mathbb{R}$
$\mathrm{Cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$
$\partial \mathbb{Q} = \mathrm{Cl}\,\mathbb{Q} \cap \mathrm{Cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$.

3. In $(\mathbb{R}^n, d)$, for $A = B_d(x, \varepsilon)$,
1. $\mathrm{Cl}\,A = \{y : d(x,y) \leq \varepsilon\}$
2. $\mathrm{Int}\,A = A$
3. $L(A) = \mathrm{Cl}\,A$ $[L(A) \subseteq \mathrm{Cl}(A)$. $\forall y \in \mathrm{Cl}\,A$, $\forall \delta$, $B(y, \delta) \cap A \setminus \{y\} \neq \phi]$
4. $\partial A = \mathrm{Cl}\,A \cap \mathrm{Cl}(\mathbb{R}^n \setminus A) = \{y : d(x,y) = \varepsilon\}$.

**Proof:** $\mathrm{Cl}(\mathbb{R}^n \setminus A) = \{y : d(x,y) \geq \varepsilon\}$: Exercise. $\subseteq$
$\supseteq$: If $y$ s.t. $d(x,y) \geq \varepsilon$, then $B(y, \delta)$
$d(x,y) > \varepsilon$, then $B(y, \delta) \cap (\mathbb{R}^n \setminus A) \neq \phi$
If $d(x,y) = \varepsilon$, then $B(y, \delta) \cap (\mathbb{R}^n \setminus A) \neq \phi$
$\Rightarrow y \in \mathrm{Cl}(\mathbb{R}^n \setminus A)$

4. In a discrete space $(X, \mathcal{T})$, $\forall A \subseteq X$,
1. $\mathrm{Cl}\,A = A$
2. $\mathrm{Int}\,A = A$
3. $L(A) = \phi$ [Given $x \in X$, $\{x\}$ is a nbhd of $x$ not in $A \setminus \{x\}$. $\Rightarrow x \notin L(A)$]
4. $\partial A = \phi$. $[\mathrm{Cl}\,A = A$, $\mathrm{Cl}(X \setminus A) = X \setminus A]$

**Defn:** A "dense set" in a topological space $(X, \mathcal{T})$ is a subset $A \subseteq X$ s.t. $\mathrm{Cl}\,A = X$ [i.e. $\forall U \in \mathcal{T}$, $A \cap U \neq \phi$].

**Eg:** 1. $\mathbb{Q}^n$ in $(\mathbb{R}^n, d)$ [any ball in $\mathbb{R}^n$ contains a point in $\mathbb{Q}^n$]
2. Let $(X, \mathcal{T})$ be a topological space where $X$ is uncountable, $\mathcal{T}$ is the cocountable topology. Then any uncountable $A \subseteq X$ is dense.
**Proof:** If $U \subseteq X$ is open, $X \setminus U$ is countable.
$\Rightarrow A \not\subseteq X \setminus U \Rightarrow A \cap U \neq \phi$.

**Def:** A top. space $(X, \mathcal{T})$ is "separable" if it has a countable dense set.
**Eg:** $\mathbb{R}^n$ with std. topology.

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Recall that a top. space $(X, \mathcal{T})$ is
(i) first countable if it has a countable basis of nbhds at every point
(ii) second countable if it has a countable basis.

Fix $(X, \mathcal{T})$ - topological space.

**Eg:** 1. Every metric space is first countable.
2. There are metric spaces that are not second countable, eg. the discrete metric on an uncountable set $X$.

1. **Def:** Let $(X, d)$ be a pseudometric space. Then the collection of sets of the form $B_d(x, \varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ where $x \in X$, $\varepsilon > 0$ is a basis for a topology on $X$, called the "pseudometric topology" $\mathcal{T}_d$.

A topology $\mathcal{T}$ on any set $Y$ is called "(pseudo)metrizable" if $\exists$ a (pseudo)metric $d : Y \times Y \to \mathbb{R}$ s.t. $\mathcal{T} = \mathcal{T}_d$

**Theorem** Every pseudo-metrizable space is first countable.

**Proof:** Left as an exercise. Recall the proof for metric spaces.
(Every metric space is first countable.
Every pseudometric space is first countable.)
**Exercise:** Every second countable space is first countable (second countable $\Rightarrow$ first countable)

**Prop:** If $(X, \mathcal{T})$ is second countable, then it is separable. (second countable $\Rightarrow$ separable)
**Proof:** If $X$ is countable, there is nothing to prove. So assume otherwise.
Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable basis for $X$.
For each $n \in \mathbb{N}$ let $x_n \in B_n$.
Then $A = \{x_1, x_2, \ldots\}$ is a dense set:
given $U \subseteq X$ open, $\exists n$ s.t. $B_n \subseteq U$ $\Rightarrow x_n \in U$
$\Rightarrow U \cap A \neq \phi$.

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## Hausdorff Spaces

**Def:** Let $(X, \mathcal{T})$ be a topological space and $x, y \in X$ be s.t. $x \neq y$.
Then $x$ and $y$ are said to be "separated by neighbourhoods" if $\exists$ nbhds $U \ni x$, $V \ni y$ s.t. $U \cap V = \phi$.
$(X, \mathcal{T})$ is said to be "Hausdorff ($T_2$)" if every pair of distinct points is separated by neighbourhoods.

**Theorem**
**Eg:** 1. All metrizable spaces are Hausdorff.
**Proof:** Given $(X, d)$ and $x \neq y$ in $X$, let $r = d(x,y)$ and $\delta = r/3$.
Then $B_d(x, \delta) \cap B_d(y, \delta) = \phi$.

2. The discrete topology is Hausdorff.

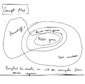
**Non-eg:** 1. $\mathbb{R}$ with cofinite topology:
for $x \neq y$, if we have nbhds $U \ni x$ and $V \ni y$ s.t. $U \cap V = \phi$,
Then $(X \setminus U) \cup (X \setminus V) = \mathbb{R}$.
This is impossible for finite $X \setminus U$, $X \setminus V$.

2. Consider the space
$$X = (-\infty, 0) \cup (0, \infty) \cup \{0_+, 0_-\}$$
whose open sets are generated by the following basis:
(*) all $(a,b) \subset (-\infty, 0)$
(*) all $(a,b) \subset (0, \infty)$
(*) all sets of the form $(-a, 0) \cup \{0_+\} \cup (0, b)$
(*) all sets of the form $(-a, 0) \cup \{0_-\} \cup (0, b)$
Then this is not a Hausdorff topology on $X$.
**Proof:** Any two neighbourhoods of the points $0^+$ and $0^-$ intersect:

**Prop:** In a Hausdorff space $(X, \mathcal{T})$, point sets $\{x\}$ are closed.
**Proof:** We need to show $U = X \setminus \{x\}$ is open.
For any $y \in U$, let $V_x$ and $V_y$ be neighbourhoods of $x$ and $y$ s.t. $V_x \cap V_y = \phi$.
$\Rightarrow V_y \subseteq U$.
$\Rightarrow U$ is open.

**Exercise:** $\exists$ spaces where all point sets are closed which are not Hausdorff. Think of an example!
**A.** Consider $(X, \mathcal{T})$ where $X$ is any uncountable set, $\mathcal{T}$ is the cocountable topology. All countable sets are closed, so any $\{x\}$ is closed. However this space is not Hausdorff. If for some $x, y$ we have open sets s.t. $V_x \cap V_y = \phi$, then $(X \setminus V_x) \cup (X \setminus V_y) = X$, but $X \setminus V_x$ and $X \setminus V_y$ are countable sets. $\Rightarrow\Leftarrow$

**Concept Map**




Throughout the semester, we will see examples from various regions.